JOURNAL OF COMBINATORIAL THEORY, Series A 60, 104–116 (1992)

Quasi-Symmetric Designs and Biplanes of Characteristic Three

S. S. SANE

Department of Mathematics, University of Bombay, Bombay-98, India

AND

M. S. SHRIKHANDE

Mathematics Department, Central Michigan University, Mt. Pleasant, Michigan 48859

Communicated by Marshall Hall, Jr.

Received March 5, 1990

Quasi-symmetric 2-designs with block intersection numbers 0 and $y \geqslant 2$ are studied using an additional property that any triple of points is contained in 0 or p blocks for some positive constant p. Some characterization results including the extremal cases $k = y^2 + y$, $\lambda = y^2 + y - 1$ and $\lambda = y^2$ are given. The paper also contains some structural investigations of biplanes of characteristic three.

INTRODUCTION

A 2-(v, k, λ) design D is called quasi-symmetric if any pair of its blocks intersects in x or y points (where x and y are non-negative numbers, $x < y$). This paper restricts itself to $(x, y) = (0, y)$. A quasi-symmetric design (q.s. design) is called proper if both the intersection numbers occur. Clearly then an improper q.s. design is just a symmetric design. In [11] Mavron and M. S. Shrikhande classified quasi-symmetric designs with $(x, y) = (0, 2)$. The main purpose of this paper is to generalize some of the results in [11] to $(x, y) = (0, y)$, $y \geqslant 2$. To achieve this, we impose an extra condition on such q.s. designs. A quasi-symmetric design is said to have property (0) if any triple of its points is contained in 0 or p blocks (where p is a positive constant). We mention that property (0) was first introduced by Cameron [6] for the restricted class of symmetric designs.

Note also that if $(x, y) = (0, 2)$ then property (0) trivially holds, since in

0097-3165/92 $3.00

that case p must be 1. Sections 1 and 3 are devoted to classification results which heavily depend on property (0). Among these are characterization results of the extremal cases $k=y^2+y$, $\lambda=y^2+y-1$ and $\lambda=y^2$ in Baartmans and M. S. Shrikhande [2], where the same results were derived under the assumption that the q.s. design contains no three mutually disjoint blocks. Section 2 is devoted to the structural investigations of biplanes of characteristic three closely following the line of arguments in [11]. In this section, we associate various designs and Steiner systems to a biplane of characteristic three which will be hopefully useful in obtaining further constraints on biplanes of characteristic three.

1. Some Classification Results on Quasi-Symmetric Designs with $x=0$

For this entire section, assume that D is a quasi-symmetric 2-(v, k, λ) *design in which any two blocks intersect in* 0 *or* $y \geqslant 2$ *points.* If both the intersection numbers occur then D will be called *proper*. Clearly then, an improper quasi-symmetric design (q.s. design) D is just a symmetric (v, k, λ)-design. The purpose of this section is to give classification results generalizing the results in Mavron and Shrikhande [11], where y was assumed to be 2. As pointed out in the Introduction, the case $y=2$ is certainly special among the class of quasi-symmetric design with $x=0$ and $y \geqslant 2$, since in that case any triple of points is contained in at most one block (else $y \geqslant 3$, a contradiction). This observation demands that we restrict our study to these D which also satisfy the following additional requirement:

Property (0). D has property (0) if there exists a constant p ($\geqslant 1$) such that every triple of points is contained in 0 or p blocks.

We record that Property (0) was introduced by Cameron [6] in the special case when D is improper, i.e., a symmetric design. Observe also that Property (0) is trivially satisfied if $y=2$ (because then $p=1$). In view of Property (0), call a *point-triple* of D *good if it is contained in some block* and bad otherwise, i.e., if no block contains it. We now repeat the following definitions borrowed from Mavron and Shrikhande [11], where $(x, y)=(0, 2)$ was assumed.

Property (I). D has Property (I) if for any given set $\{q_1, q_2, q_3, q_4\}$ of four points the conclusion (q_2, q_3, q_4) is a good triple follows from the hypothesis: (q_1, q_2, q_4) and (q_1, q_3, q_4) are both good triples.

Property (II). D has Property (II) for a non-flag (s, B) if given a set $\{q_1, q_2, q_3\}$ of three points contained in B; the conclusion (q_2, q_3, s) is a good triple follows from the hypothesis: (q_1, q_2, s) and (q_1, q_3, s) are both good triples.

Property (III). D has Property (III) for a non-flag (s, B) if given a set $\{q_1, q_2, q_3\}$ of three points contained in B, the conclusion (q_2, q_3, s) is a bad triple follows from the hypothesis: (q_1, q_2, s) and (q_1, q_3, s) are both bad triples.

With certain exceptions, it turns out that D satisfying Property (0) and one of Property (I) or Property (II) is actually a 3-design of the following type C:

DEFINITION 1.1. A quasi-symmetric 3-design D is called type C if it is a 3-(v, k, λ) design where $v=(\lambda+1)(\lambda^2+5\lambda+5)$, $k=(\lambda+1)(\lambda+2)$, $\lambda=\lambda_3$, $x=0$, $y=\lambda+1$.

THEOREM 1.2 (Cameron [4]). *Let D be a* 3-(v, k, λ) *design such that D is an extension of a symmetric design. Then $x=0$ and conversely. Moreover, D is either a Hadamard* 3-*design, a type C design, a* 3-(496, 40, 3) *quasi-symmetric design with $x=0$ and $y=4$ or a* 3-(112, 12, 1) *quasi-symmetric design with $x=0$ and $y=2$.*

THEOREM 1.3 (Lam *et al.* [9]). *The last possibility in Theorem* 1.2 *does not exist.*

Of the properties stated here, Property (III) has not yielded to a generalization ($y \geqslant 3$). We therefore record

THEOREM 1.4 (Mavron and Shrikhande [11, Theorem 3.8]). *Let $y=2$ and suppose D has Property* (III). *Then D is the extension of the projective plane of order* 2 *or* 4 *or D is the unique biplane with $k=4$* (*symmetric* (7, 4, 2)-*design*) *or the unique* 2-(21, 6, 4) *q.s. design.*

LEMMA 1.5 [16]. *The following equation is satisfied by the parameters of D*:

$$(r-1)(y-1)=(k-1)(\lambda-1).$$

LEMMA 1.6. *Let $v-1>k>y+1$. Then $k \geqslant \lambda+1$ with equality if and only if D is an extension of a symmetric design* (*and then the parameters of D are as given in Theorem* 1.2).

Proof. A repetition of the argument in [11, Lemma 2.2] proves this: If q is any point then D_q the derived design at q is a 1-design whose dual is a 2-design. Application of Fisher inequality to that design gives $k \geqslant \lambda+1$

with equality if and only if the dual of D_q and hence D_q is a symmetric design. (Actually, the standard parameter relations for the same design prove Lemma 1.5.)

THEOREM 1.7. *Let D be a q.s. 2-design with $x=0$ and $y \geqslant 2$ such that D has both Properties* (0) *and* (I). *Then D is an extension of a symmetric design* (*and conversely*).

Proof. This is essentially contained in [11, Theorem 3.2]. Let (q_1, q_2, q_3) be any point-triple. Using Property (0), it is enough to show that this triple is contained in some block, which will prove that D is a 3-(v, k, λ_3) design with $\lambda_3 = p$ (and $x=0$) and our proof will be complete in view of Theorem 1.2.

Let B_{12} and B_{13} be two blocks containing (q_1, q_2) and (q_1, q_3), respectively. Since $y \geqslant 2$ these blocks intersect in some point $s \neq q_1$. Then (s, q_2, q_1) and (s, q_3, q_1) are both good triples. So Property (I) implies that (q_2, q_3, q_1) is a good triple. Hence the proof.

The following result extends [11, Theorem 3.4].

THEOREM 1.8. *Let D be a q.s. design with $x=0$ and $y \geqslant 2$. Assume that D has Property* (0) *and Property* (II) *for some non-flag* (s, B). *Then D is one of the following four types*:

(a) *$y=2$ and the unique* 3-(8, 4, 1), *i.e., an extension of a projective plane of order two.*

(b) *D is a biplane, i.e., a symmetric* $(v, k, 2)$-*design, and if D has characteristic* (*see* [5] *or Section* 2 *of the present paper*) *then the characteristic is three.*

(c) *D is an extension of a symmetric design*; *i.e., D is a* 3-*design with* $x=0$.

(d) *D is a* 2-(100, 12, 5)-*design with* $y=2$.

Proof. If $y=2$ then [11, Theorem 3.4] gives (a), (b), and (d) as the only possibilities for D. So let $y \geqslant 3$ where, by Theorem 1.2, it suffices to show that D is a 3-design in order to conclude that D is as in (c). Our proof is similar to the one in [11, Theorem 3.4]. Define a relation $\sim$ on the point-set of B by writing $q_1 \sim q_2$ if (s, q_1, q_2) is a good triple. Then Property II implies that $\sim$ is an equivalence relation on the point-set of B. Using Property (0), the size of each equivalence class is $1+\lambda(y-1)/p$, where p is the constant involved in Property (0) (make a two-way counting of pairs (q, q') where q is a fixed point of B, $q' \in B$ such that (s, q, q') is a good triple). Let u be any point of D and consider the derived design D_u at u. As in the proof of Lemma 1.6, the dual D_u^* of D_u is a 2-design and by Property (0), any point-pair occurs in 0 or p blocks. Therefore D_u^* is a quasi-symmetric 2-design with parameters $v^* = r$, $b^* = v - 1$, $r^* = k - 1$, $k^* = \lambda$, $\lambda^* = y - 1$, $x^* = 0$, and $y^* = p$. By Lemma 1.5 (applied to D_u^*) we have $(r^* - 1)(y^* - 1) = (k^* - 1)(\lambda^* - 1)$, i.e.,

$$(k-2)(p-1) = (\lambda - 1)(y-2). \tag{1}$$

Since D is itself a q.s. design, Lemma 1.5 (applied to D) again gives

$$(r-1)(y-1) = (k-1)(\lambda - 1). \tag{2}$$

Now assume that we have at least two equivalence classes under $\sim$. Then $2(1 + \lambda(y-1)/p \leqslant k$, which implies that $k - 2 \geqslant \lambda y + y - 2$. Since $y \geqslant 3$ (and $\lambda \geqslant 3$), $\lambda y + y - 2 > (\lambda - 1)(y - 2)$. So $k - 2 > (\lambda - 1)(y - 2)$, which by (1) implies that $p = 1$. But then $y \leqslant 2$, a contradiction. So there is only one equivalence class under $\sim$.

Therefore the size of an equivalence class $= 1 + \lambda(y-1)/p = k$, i.e., $p(k-1) = \lambda(y-1)$. Subtracting (1) from this equation obtains $p + k = \lambda + y$, i.e., $p + (k-1) = \lambda + (y-1)$ and $p(k-1) = \lambda(y-1)$. From these two equations $p = \lambda$ and $k - 1 = y - 1$ or $p = y - 1$ and $k - 1 = \lambda$. Of these, the first possibility cannot occur since $y \neq k$ (for, $y = k$ by Lemma 1.5 implies $r = \lambda$ which gives $v = k$, a contradiction). So $\lambda = k - 1$ and Lemma 1.6 proves the result.

Let D be a *q.s. design with* $y = 2$. An arc in D is a set A of points of D with the property that no three points of A are contained in a block of D, i.e., A contains no good triple. The following is a particular case of a result in [17].

LEMMA 1.8. *With everything as above let A be arc of D. Then $|A| \leqslant r/\lambda + 1$ with equality if and only if every block meets A in 0 or 2 points.*

Proof. For a fixed point q in A count flags of the type (z, Z), where q, z are in Z and $z \in A$. Then a two-way counting shows that $(|A| - 1)\,\lambda \leqslant r$. Hence the inequality. Clearly equality holds if and only if every Z containing z contributes precisely one point to the set of such flags, i.e., $|Z \cap A| = 2$ for all the blocks Z containing z.

DEFINITION 1.9. An arc A satisfying equality in Lemma 1.8 (i.e., $|A| = 1 + r/\lambda$) is called a *maximal arc* (in a q.s. design with $(x, y) = (0, 2)$).

THEOREM 1.10. *Let D be a q.s. design with $(x, y) = (0, 2)$. Suppose D has a maximal arc. Then one of the following holds:*

(i) *D is a biplane with k even.*

(ii) *D is a proper quasi-symmetric design with* $v=(s+1)^2(s^2+3s+1)+1$, $k=(s+1)(s+2)$, *and* $\lambda=s+3$, $s\geqslant 1$.

Proof. By Lemma 1.8, λ divides r and by Lemma 1.5, $r=(k-1)(\lambda-1)+1$. Hence λ divides $k-2$. Write $k=s\lambda+2$, $s\geqslant 1$. First assume that $s=1$. Then $k=\lambda+2$ and therefore, for a non-flag (s, B), given $q\in B$ the number of points w on B such that (s, q, w) is a good triple is exactly λ (because $y=2$). So there is a unique point q' on B for which (s, q, q') is a bad triple. It is then clear that Property (III) is vacuously satisfied. By using [11, Theorem 3.8] and noting that D is not a 3-design (because there are bad triples), D is either the unique biplane with $k=4$ or the 2-(21, 6, 4) design.

Now assume that $s\geqslant 2$. By Lemma 1.5, $\lambda\geqslant y=2$ with equality implying that D is symmetric; i.e., D is a biplane. Since $k=s\lambda+2$, $\lambda=2$ gives D as in (i). Let $\lambda\geqslant 3$ and $s\geqslant 2$. Then $r=(k-1)(\lambda-1)+1=(s\lambda-1)(\lambda-1)+1$. Using $\lambda(v-1)=r(k-1)$ and $k=s\lambda+2$, we obtain $v=(s(\lambda-1)+1)(s\lambda+1)+1$. Since $vr=bk$, we obtain $(s+2)(\lambda-2)\equiv 0 \pmod{s\lambda+2}$, where $\lambda\geqslant 3$. Hence $s\lambda+2\leqslant(s+2)(\lambda-2)$ which implies $\lambda\geqslant s+3$. But then $(s+2)(\lambda-2)\equiv 2(\lambda-(s+3)) \pmod{s\lambda+2}$ and therefore $s\lambda+2$ divides $2(\lambda-(s+3))$. If $\lambda\neq s+3$ then $s\lambda+2\leqslant 2(\lambda-(s+3))$ gives $(s-2)(\lambda+2)\leqslant -12$ which is absurd. So $\lambda=s+3$ and the parameters of D are as given in (ii).

Remark 1.11. Taking $s=1$ in Theorem 1.10 obtains the unique 2-(21, 6, 4) quasi-symmetric design. The case $s=2$ gives the parametrically feasible 2-(100, 12, 5) q.s. design. Both 2-(21, 6, 4) and 2-(100, 12, 5) designs arise in [11, Theorems 3.6 and 3.8].

2. Biplanes of Characteristic Three

A subclass of the larger class of quasi-symmetric designs with $(x, y)=(0, 2)$ studied in [11] is the family of biplanes, i.e., symmetric $(v, k, 2)$ designs. No example of a biplane with $k\geqslant 14$ seems to be known, nor is it known whether there are only finitely many biplanes. Hussain in [10] associated a chain structure (graph) with every non-flag (z, B) of a biplane D in the following manner: $G=G(z, B)$ is the graph with vertex-set the point-set of B and two points z_1, z_2 adjacent if (z, z_1, z_2) is a good triple. It is easily seen that G is a disjoint union of cycles. Following Cameron [5], if $G(z, B)$ is a disjoint union of m-cycles for every non-flag (z, B) and m is a constant independent of (z, B) then D is said to have characteristic m. Biplanes of characteristic three exist for $k=3$ and 6 and the next feasible value is $k=18$ as shown in

THEOREM 2.1 [11]. *If a biplane has characteristic three then* $k \equiv 0 \pmod 3$, *v is even and $k-2$ is a perfect square.*

Observe that if (z, B) is a non-flag of a biplane then the graph $G(B, z)$ defined on the blocks containing z in a dual manner is isomorphic to the graph $G(z, B)$ under the natural correspondence between the blocks on z and the point-pairs (edges in $G(z, B)$ in B. Hence the dual of a biplane of characteristic three is a biplane of characteristic three. Theorem 2.1 was proved in [11] using somewhat involved counting argument. The purpose of the present section is to break that argument in parts (called assertions) and use these assertions to obtain stronger structural information which would hopefully be useful in pinning down the structure of a biplane of characteristic three. However, we confess that our treatment has failed to improve the numerical constraints imposed on the parameters of a biplane of characteristic three given in Theorem 2.1. We begin by noting an obvious special case of Theorem 1.8: Let D be a biplane. Then D has characteristic three if and only if Property (II) holds for every non-flag.

From this point on, assume that D is a biplane of characteristic three.

PROPOSITION 2.2. *Fix a block B. For every $z \notin B$, define $D(z, B)$ to be the set of $k/3$ subsets of size 3 which are 3-cycles (treated as point-sets in B) in $G(z, B)$. Form an incidence structure D^* whose point-set is B and whose set of blocks is the union of $D(z, B)$ for all $z \notin B$. Then D^* is a resolvable 2-design with parameters $v^* = k$, $b^* = (v-k)\,k/3$, $r^* = v-k$, $k^* = 3$, and $\lambda^* = k-2$, such that D^* has no repeated blocks.*

Proof. This is obvious; every point $z \notin B$ gives rise to a resolution. Also, if z_1, z_2 determine the same triple in B then we get two blocks meeting in z_1, z_2, and a point of B—a contradiction.

PROPOSITION 2.3. *Fix two blocks X and Y intersecting in points α and β. For every $z \notin Y$, $z \neq \alpha, \beta$ let $D(z, X)$ be defined exactly as in Proposition 2.2; i.e., $D(z, X)$ consists of the $k/3$ subsets of size 3 given by the 3-cycles in $G(z, X)$. Out of these subsets precisely one contains α and β. Define $D_1(z, X)$ to be the subset of $D(z, X)$ obtained by deleting the subset containing α and β (then $D_1(z, X)$ consists of $k/3-1$ triples). Define the incidence structure D_1 whose point-set is $X - \{\alpha, \beta\}$ and whose block-set is the union of $D_1(z, X)$ for all $z \in Y$, $z \neq \alpha, \beta$. Then D_1 is a 2-design without repeated blocks and with parameters: $v_1 = k-2$, $b_1 = (k-2)(k-3)/3$, $r_1 = k-3$, $k_1 = 3$, and $\lambda_1 = 2$. Further, the blocks of D_1 can be partitioned into $k-2$ sets, each set with $(k-3)/3$ pair disjoint blocks (and one point not covered).*

Proof. A straightforward verification; every class corresponds to a point of Y other than α, β. For a point-pair in $X - \{\alpha, \beta\}$, the unique block containing that pair intersects Y in two points. Hence $\lambda_1 = 2$.

We now outline the main points (from [11]) used in the proof of Theorem 2.1. These are called assertions and their proofs can be found in [11]. Our setup is as follows: X and Y are two blocks intersecting in points α, β. Write $X' = X - \{\alpha, \beta\}$, $Y' = Y - \{\alpha, \beta\}$. In general if $T = \{a_1, a_2, ..., a_m\}$ is a set of at least three points contained in a block B then B is uniquely determined by this set of m points. In that case, write $B = \langle a_1, a_2, ..., a_m \rangle$. Let Z be a block intersecting X' in x_1, x_2, $Z \neq X$. Suppose $Z \cap Y = \{y_1, y_2\}$. We then have $y_1, y_2 \in Y$ and $Z = \langle x_1, x_2, y_1 \rangle = \langle x_1, x_2, y_2 \rangle = \langle x_1, y_1, y_2 \rangle = \langle x_2, y_1, y_2 \rangle$.

ASSERTION 1. *Let $W \neq Z$ be the block containing x_1 and y_1. Suppose $W \cap X = \{x_1, x_3\}$ and $W \cap Y = \{y_1, y_3\}$. Then the points x_1, y_1, x_3, y_3 are all distinct.*

As already remarked, the proof of Assertion 1 and all the other assertions to follow are essentially given in [11].

ASSERTION 2. *Let $W_i \neq Z$ be two blocks such that W_i contains x_i and y_i, $i = 1, 2$. Suppose $W_1 \cap X = \{x_1, x_3\}$, $W_i \cap Y = \{y_1, y_3\}$, $W_2 \cap X = \{x_2, x_4\}$, and $W_2 \cap Y = \{y_2, y_4\}$. Then $x_3 \neq x_4$ and $y_3 \neq y_4$.*

For the sake of a variation, we prove Assertion 2: (x_3, Z) is a nonflag, $Z = \langle x_1, x_2, y_1 \rangle$, and (x_3, x_1, x_2), (x_3, x_1, y_1) are both good triples. So (x_3, x_2, y_1) is a good triple. Similarly, (x_4, x_1, y_2) is a good triple. If $x_3 = x_4$ then the unique block containing x_3, x_1, y_2 must be W_1, since $y_2 \notin X$. So $W_1 = \langle x_3, x_1, y_2 \rangle$. But W_1 contains y_1 and hence $W_1 = \langle x_3, x_1, y_1, y_2 \rangle$. But $\langle x_1, y_1, y_2 \rangle = Z$ implies $W_1 = Z$, a contradiction. Similarly $y_3 \neq y_4$.

ASSERTION 3. *Define two more blocks U_1, containing x_1 and y_2, and U_2 containing x_2 and y_1, where U_1, $U_2 \neq Z$. Then $U_1 = \langle x_1, y_2, x_4, y_3 \rangle$ and $U_2 = \langle x_2, y_1, x_3, y_4 \rangle$.*

Proof. (y_3, Z) is a non-flag, $Z = \langle y_1, y_2, x_1 \rangle$, and (y_3, y_1, y_2), (y_3, y_1, x_1) are both good. So (y_3, y_2, x_1) is a good triple. But $y_3 \notin Z$. So $\langle y_3, y_2, x_1 \rangle = U_1$. Other parts are similar.

ASSERTION 4. *$\langle x_3, x_4, y_3, y_4 \rangle = Z'$ for some Z'; i.e., there is a block Z' containing x_3, x_4, y_3, y_4.*

In [11], the correspondence between Z and Z' (which is also obtained as a correspondence between the point-pairs of X') is used to show that the number of blocks of D not containing α, β is an even number; i.e., $(k-2)(k-3)/2$ is an even number. This at once shows that v is even and then $k-2$ must be a perfect square follows from the Bruck–Ryser–Chowla (see [3]) conditions. Implicit in that argument is:

ASSERTION 5. *Let S be the set of all the blocks of D not containing α, β. For $Z \in S$ and Z described before Assertion* 1, *define $M \sim Z$ if M is any one W_1, W_2, U_1, U_2, Z, Z'. Then $\sim$ is an equivalence relation on S.*

Essentially, Assertion 5 says that Z can be replaced by any one of Z', U_1, U_2, W_1, W_2 and we still obtain the same subset of six blocks if they are defined exactly as in Assertions 1 through 4. There are five nice consequences of Assertions 1 through 5 and these are listed in the following theorem.

THEOREM 2.4. *Let X, Y, α, β, S be defined as above Then*:

(i) *$|S|$ is divisible by* 6, *i.e.*, $(k-2)(k-3)/2 \equiv 0 \pmod 6$. *Since* $k \equiv 0 \pmod 3$, *we must have* $k \equiv 3$ *or* $6 \pmod{12}$. (*Observe that this also proves Theorem* 2.1).

(ii) *The equivalence relation $\sim$ defined on S also defines an equivalence relation $\sim$ on the set Q of all the point-pairs of $X' = X - \{\alpha, \beta\}$ under the one-to-one correspondence between blocks in S and members of Q. A typical equivalence class in Q consists of all the* 6 *point-pairs of a* 4-*subset $\{x_1, x_2, x_3, x_4\}$ of X'. For example, the equivalence class $\{Z, Z', W_1, W_2, U_1, U_2\}$ in S induces $\{(x_1, x_2), (x_3, x_4), (x_1, x_3), (x_2, x_4), (x_1, x_4), (x_2, x_3)\}$.*

(iii) *For an equivalence class E in Q as described in* (ii) *let E_s be the set* (*support*) *of four points of X' which occur in members of E. Define an incidence structure D_2 whose point-set is X' and whose blocks are E_s, where E is an equivalence class in Q. Then D_2 is a Steiner system with parameters $v_2 = k-2$, $k_2 = 4$, and $\lambda_2 = 1$.*

(iv) *If D_2^* is a Steiner system* 2-$(k-2, 4, 1)$ *defined on the point-set Y' in the same manner D_2 is defined in* (iii) *then there is a one-to-one correspondence between blocks of D_2 and D_2^*: in the notation of Assertions* 1 *through* 5 *$\{y_1, y_2, y_3, y_4\}$ corresponds to $\{x_1, x_2, x_3, x_4\}$ and both are induced by $\{Z, Z', W_1, W_2, U_1, U_2\}$.*

(v) *Recall the definition of the design D_1 given in Proposition* 2.2. *Then every block of D_1* (*whose size is three*) *is contained in a block of D_2. In fact, the blocks of D_1 are precisely obtained by taking all the four* 3-*subsets of a block of D_2 for every block of D_2.*

Proof. All the assertions follow from our earlier assertions 1 through 5. To prove (v), take a typical block $\{x_1, x_2, x_3, x_4\}$ of D_2 and observe that in D_1, $\{x_1, x_2, x_3\}$ is induced by $y_1 \in Y'$ and other triples are similarly induced by y_2, y_3, y_4.

3. Further Results on Quasi-Symmetric Designs with $x=0$

In this final section, we prove some results on quasi-symmetric designs with $x=0$. A major part in this section revolves around Cameron's theorem (Theorem 1.2) and in particular on characterizations of designs of type C. The two basic results we use are:

Lemma 3.1. *Let D be a quasi-symmetric 2-(v, k, λ) design with $x=0$ and $y \geqslant 2$ such that D satisfies Property* (0) *(for some p). Suppose D is not a 3-design and let u be any point of D. If D_u^* denotes the dual of D_u, the derived design at u then D_u^* is a proper q.s. design with $k^*=\lambda$, $x^*=0$, and $y^*=p$. Hence p divides λ.*

Proof. This has already been shown in the proof of Theorem 1.8. If D_u^* is symmetric then so is D_u and then D is a 3-design. By a well-known result, (see, e.g., [16]) y^* divides k^* in a proper q.s. design.

Theorem 3.2 [16]. *Let D be a proper quasi-symmetric 2-(v, k, λ) design with $x=0$ and $y \geqslant 2$. Let m be the integer k/y (see [16]) and for two given disjoint blocks let $\bar{c}$ be the number of blocks disjoint from both of them (this is an invariant of D). Then λ satisfies the quadratic $A\lambda^2+B\lambda+C=0$, where A, B, C are polynomial functions of m, y, and $\bar{c}$ whose values are given by: $A=(m-1)\,y[m(y+1)-m^2-1]$, $B=-[(m-1)\,y\{2m(y^2+1)-2m^2y-(y+1)\}-(y-1)^2-my\,\bar{c}]$, and $C=-(my-1)\,y^2(m-1)^2$.*

The special case $\bar{c}=0$ of Theorem 3.2 was considered in [2], where it was proved that $2 \leqslant m \leqslant y+1$ and it was also shown that the extremal case $m=y+1$ can be characterized as a design of type C or its quasi-residual. This section attempts to remove the condition $\bar{c}=0$ (in extremal cases) with partial success.

Theorem 3.3. *Let D be a proper quasi-symmetric 2-(v, k, λ) design with $x=0$ and $y \geqslant 2$ such that $k=y^2+y$. Then $\beta=(\lambda-1)/(y-1)$ is an integer and*

$$(y+2-\beta)(y+1-\beta)=\frac{(y+1)\,\bar{c}\lambda}{y} \tag{3}$$

holds, where $\bar{c}$ has the same meaning as in Theorem 3.1. Also the following assertions hold:

(i) *$\beta \leqslant y+2$ with equality if and only if D is a design of type C.*

(ii) *$\beta=y+1$ if and only if D is a quasi-residual of a design of type C.*

(iii) *If D satisfies Property* (0) *then D is a design of type C or the unique 2-$(21, 6, 4)$ design with $(x, y)=(0, 2)$.*

Proof. Since $k=y^2+y$, Lemma 1.5 implies that $y-1$ divides $\lambda-1$. In the quadratic of Theorem 3.2, write $\lambda=\beta(y-1)+1$, $m=y+1$ (since $k=y(y+1)$ and simplify to obtain (3). Lemma 1.6 implies that $\beta \leqslant y+2$ with equality if and only if D is a 3-design. Then in (3), $\bar{c}=0$ and (i) a direct consequence of [2, Theorem 3.6]. Alternatively observe that D must be an extension of a symmetric design and Theorem 1.2 tells us that $k=y^2+y$ if and only if D has type C. This proves (i). For (ii) again observe that $\beta=y+1$ yields $\bar{c}=0$ and [2, Proposition 3.8] is applicable.

Finally consider (iii). Then with the substitution $\lambda-1=\beta(y-1)$ and $k=y^2+y$, (1) reads: $(y+2)(p-1)=\beta(y-2)$. If $y=2$ or $p=1$ then (i) implies that $\beta=3$ or 4 and hence $\bar{c}=0$. So [2, Theorem 3.6 and Proposition 3.8] are applicable again to show that D is the 3-(22, 6, 1) design or its residual, the 2-(21, 6, 4) design. Hence assume that $y \geqslant 3$. If y is odd then $y-2$ and $y+2$ are coprime and therefore $y+2$ divides β, i.e., $y+2 \leqslant \beta$ and we are done by (i). If y is even then the g.c.d. of $y+2$ and $y-2$ divides 4.

So $y+2$ divides $4\beta \leqslant 4(y+2)$. Hence β equals $y+2$, $(y+2)/2$, $3(y+2)/4$, or $(y+2)/4$. In the first case we are done by (i). In the second case, use (3) to obtain $y+1$ divides 1, a contradiction. In the third and the fourth cases, (3) yields $y+1$ divides 3, i.e., $y=2$, contrary to the assumption. This completes the proof of Theorem 3.3.

THEOREM 3.4. *Let D be a quasi-symmetric 2-(v, k, λ) design with $x=0$ and $y \geqslant 2$. Suppose $\lambda=y^2$. Then D is a residual of a design of type C.*

Proof. In view of Theorem 3.3 (ii), it is enough to show that $k=y^2+y$. Write $k=my$ where m is an integer, use Lemma 1.5 and substitute $\lambda=y^2$ to obtain $r=my^2+(m-1)y$. Next $\lambda(v-1)=r(k-1)$ gives $v-1=(my+m-1)(my-1)/y$ whence y divides $m-1$. Write $m=\alpha y+1$ to obtain $r=y^2(\alpha y+\alpha-1)$ and $v=(\alpha y+\alpha+1)(\alpha y^2+y-1)+1$. Using $bk=vr$, we obtain $k=my$ divides vr and hence $m=\alpha y+1$ divides $[(\alpha y+\alpha+1)(\alpha y^2+y-1)+1]\cdot(\alpha y+\alpha+1)y$. Therefore, $\alpha-1 \equiv 0 \pmod{\alpha y+1}$. So either $\alpha=1$ or $\alpha y+1 \leqslant \alpha-1$ which is absurd. Hence $\alpha=1$ and $m=y+1$; i.e., $k=y^2+y$ and we are finished.

THEOREM 3.5. *Suppose D is a quasi-symmetric 2-(v, k, λ) design with $x=0$ and $y \geqslant 2$. Let $\lambda=y^2+y-1$ and assume that D satisfies Property* (0). *Then either*

(i) *D is a 3-design of type C or*

(ii) $v=100$, $k=12$, $\lambda=5$, *and* $y=2$.

Proof. If D is a 3-design then by Lemma 1.6, $k=y^2+y=\lambda+1$ and by Theorem 1.2, D must be of type C. So assume that *D is not a 3-design.*

Clearly $p=1$ if and only if $y=2$ and this case is treated separately. So assume further that $y\neq 2$ and $p\neq 1$. We show that this leads to a contradiction.

In Eq. (1) substitute the value of λ to obtain $(k-2)(p-1)=(y+2)(y-1)(y-2)$. Since y divides k we must have $2(p+1)\equiv 0 \pmod{y}$. Lemma 1.6 and Eq. (1) imply that $p+1<y$ and y divides $2(p+1)$. So $y=2(p+1)$. Hence $\lambda=4(p+1)^2+2(p+1)-1$. Lemma 3.1 then implies that p divides λ, i.e., p divides 5. Therefore $p=5$ (since $p\neq 1$). This gives $y=12$, $\lambda=155$, and $k=387$ (using Eq. (1)). So $y=12$ must divide $k=387$, a contradiction.

We are finally left with $p=1$ and $y=2$. Then $\lambda=y^2+y-1=5$. So Lemma 1.5 gives $r=4k-3$. Hence $v-1=(4k-3)(k-1)/5$ and 5 divides $(k-2)(k-1)$. Compute $b=vr/k$ to obtain $[(4k-3)(k-1)+5](4k-3)\equiv 0 \pmod{k}$, i.e., k divides 24. Since $k>\lambda+1$ (by Lemma 1.6 and D is not a 3-design), we have $k=8$, 12, or 24. But 5 divides $(k-2)(k-1)$ and therefore $k=12$ is the only possibility.

Acknowledgment

The first author gratefully acknowledges the hospitality of the Mathematics Department, Central Michigan University while doing this work.

References

1. E. F. Assmus and J. H. van Lint, Ovals in projective designs, *J. Combin. Theory Ser. A* **27** (1979), 307–324.
2. A. Baartmans and M. S. Shrikhande, Designs with no three mutually disjoint blocks, *Discrete Math.* **40** (1982), 129–139.
3. T. Beth, D. Jungnickel, and H. Lenz, "Design Theory," Wissenschaftverlag, Mannheim, 1985 and Cambridge Univ. Press, Cambridge, UK, 1986.
4. P. J. Cameron, Extending symmetric designs, *J. Combin. Theory Ser. A* **14** (1973), 215–220.
5. P. J. Cameron, Biplanes, *Math Z.* **131** (1973), 85–101.
6. P. J. Cameron, Near regularity conditions for designs, *Geom. Dedicata* **1** (1973), 213–223.
7. P. J. Cameron and J. H. van Lint, "Graphs, Codes and Designs," London Math. Soc. Lecture Note Series, Vol. 43, Cambridge Univ. Press, Cambridge, 1980.
8. W. Haemers, Regular two graphs and extensions of partial geometries, preprint.
9. C. W. H. Lam, L. Thiel, S. Swiercz, and J. McKay, The non-existence of ovals in the projective plane of order 10, *Discrete Math.* **45** (1983), 319–321.
10. Q. M. Hussain, Symmetrical incomplete block designs with $\lambda=2$, $k=8$, or 9, *Bull. Calcutta Math. Soc.* **37** (1945), 115–123.
11. V. C. Mavron and M. S. Shrikhande, On designs with intersection numbers 0 and 2, *Arch. Math.* **52** (1989), 407–412.
12. T. P. McDonough and V. C. Mavron, Symmetric designs and geometroids, *Combinatorica* **9**, No. 1 (1989), 51–57.

13. R. M. PAWALE, Quasi-symmetric 3-designs with triangle-free graph, *Geom. Dedicata* **37**, No. 2 (1991), 205–210.
14. S. S. SANE, On extendable planes of order 10, *J. Combin. Theory Ser. A* **38** (1985), 91–93.
15. S. S. SANE, Hussain chains revisited, *Discrete Math.* **70** (1988), 211–213.
16. S. S. SANE AND M. S. SHRIKHANDE, Finiteness questions in quasi-symmetric designs, *J. Combin. Theory Ser. A* **42** (1986), 252–258.
17. S. S. SANE, S. S. SHRIKHANDE, AND N. M. SINGHI, Maximal arcs in designs, *Graphs Combin.* **1** (1985), 97–106.
18. M. S. SHRIKHANDE, Quasi-symmetric designs and quadratic polynomials, *in* "Colloquia Mathematica Societatis János Bolyai, Vol. 52," pp. 471–480, North Holland, Amsterdam, 1987.
19. M. S. SHRIKHANDE AND S. S. SANE, "Quasi-Symmetric Designs," London Math. Soc. Lecture Note Series, Vol. 164, Cambridge Univ. Press, Cambridge, 1991.